

Теоремы типа Бохера для динамических уравнений на временных шкалах

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Аннотация. Найдены условия, при которых все решения систем динамических уравнений на временных шкалах стремятся к конечным пределам при $t \rightarrow \infty$.

Theorems of Bôcher's type for dynamic equations on time scales

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Abstract

The conditions are found that all solutions of a systems dynamic equations on time scales tends to finite limits as $t \rightarrow \infty$.

A. Wintner [1,2] has named by theorems of type Bôcher a assertions which is guaranteed that the each nontrivial solution of the linear system of ordinary differential equations

$$\frac{dx}{dt} = A(t)x, \quad t_0 \leq t < \infty,$$

where $A(t)$ is $n \times n$ matrix, tends to nontrivial finite limit as $t \rightarrow \infty$. It has place, if the elements of matrix $A(t)$ belong to $L_1[t_0, \infty]$. This result is connected with name of Bôcher. [3] (see also [4]). A. Wintner has receive series of less restrictive conditions. P. Hartman [5] has establish analogue of Bôcher's result for nonlinear system.

In this paper analogical problem is studied for dynamical equations on time scales.

We consider the linear equation

$$x^\Delta = A(t)x, \quad t \in T, \tag{1}$$

where $T \subset \mathcal{R}$ is time scale and $\sup T = \infty$, $A(t)$ be an $n \times n$ -valued function on T . It is assumed that $A(t)$ is rd-continuous and regressive (see [6]).

The system(1) will be said to be of class (S) if (i) every solution $x(t)$ of (1) has a limit x_∞ as $t \rightarrow \infty$, and (ii) for every constant vector x_∞ there is a solution $x(t)$ of (1) such that $x(t) \rightarrow x_\infty$ as $t \rightarrow \infty$.

Evidently that (1) is of class (S) if and only if for every fundamental matrix $X(t)$ of (1), $X_\infty = \lim X(t)$ exists as $t \rightarrow \infty$ and is nonsingular.

Let's note still that (1) is of class (S), if it has a fundamental matrix of form

$$X_0(t) = I + o(1)$$

as $t \rightarrow \infty$.

Let $a \in T$, $a > 0$. Let integral $\int_a^\infty A(s)ds$ is convergent absolutely. Following theorem well known.

Theorem 1. *If integral $\int_a^\infty A(s)\Delta s$ is convergent absolutely. Then system (1) is of class (S).*

Proof. The solution of system (1) are represented in the form

$$x(t) = x(a) + \int_a^t A(s)x(s)\Delta s. \quad (2)$$

We have the inequality

$$|x(t)| \leq |x(a)| + \int_a^t |A(s)||x(s)|\Delta s. \quad (3)$$

The Gronwall's inequality for times scales gives

$$|x(t)| \leq |x(0)|e_{|A|}(t, a), \quad t \in T,$$

where $e_{|A|}(t, a)$ is exponential function of equation

$$x^\Delta = |A(t)|x.$$

Consequently all solutions of system (1) is bounded. Let $t_2 > t_1$. Let's evaluate norm $|x(t_2) - x(t_1)|$:

$$|x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} |A(s)||x(s)|\Delta s. \quad (4)$$

From inequality (4) follows that $|x(t_2) - x(t_1)|$ can be made as much as small if to take t_1 big enough, Therefore $|x(t)|$ converges to the finite limits as $t \rightarrow \infty$. From Gronwall's inequality follows

$$|x(t)| \geq |x(s)|(e_{|A|}(t, s))^{-1}, \quad t, s \in T. \quad (5)$$

The inequality (5) implies that $x_\infty \neq 0$ unless $x(n) \equiv 0$. This proves theorem 1.

Let now integral $\int_a^\infty A(s)\Delta s$ is convergent (possibly just conditionally). In system (1) make the change of variables

$$x(t) = y(t) + Y(t)y(t),$$

where $N \times N$ matrix $Y(t)$ we shall choose later. We obtain

$$[I + Y(\sigma(t))]y^\Delta + Y^\Delta(t)y(t) = A(t)y(t) + A(t)Y(t)y(t). \quad (6)$$

Let

$$Y^\Delta(t) = A(t).$$

Then

$$Y(t) = - \int_t^\infty A(s)\Delta s.$$

Evidently $Y(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore matrix $I + Y(\sigma(t))$ has a bounded converse for sufficiently large t . Therefore we obtain for sufficiently large t

$$y^\Delta(t) = (I + Y(t))^{-1}A(t)Y(t)y(t). \quad (7)$$

Theorem 2. *Let integral $\int_a^\infty A(s)\Delta s$ is convergent and integral*

$$\int_a^\infty A(t) \left(\int_{s=t}^\infty A(s)\Delta s \right) \Delta t \quad (8)$$

is convergent absolutely. Then system (1) is of class (S).

Proof. From conditions theorem 2 follows that right part of system (7) satisfy conditions of theorem 1.

Remark. If $X(t)$ is fundamental matrix of system (1), then $X^{-1}(t)$ is fundamental matrix of the system

$$x^\Delta = x'(\ominus A)(t),$$

where (see [6])

$$(\ominus A)(t) = -A(t)[I + \mu(t)A(t)]^{-1}.$$

Therefore the theorem 2 remains fair, if the requirement of absolute convergence integral (8) to replace with the requirement of absolute convergence of integral

$$\int_a^\infty \left(\int_{s=t}^\infty (\ominus A)(s)\Delta s \right) A(t)\Delta t.$$

Further we receive the theorem.

Theorem 3. *Let integrals $\int_a^\infty A(s)\Delta s$ and $\int_a^\infty A(t) \left(\int_{s=t}^\infty A(s)\Delta s \right) \Delta t$ are convergent and integral*

$$\int_a^\infty A(t) \left(\int_{s=t}^\infty A(s) \left(\int_{\tau=s}^\infty A(\tau)\Delta \tau \right) \Delta s \right) \Delta t$$

is convergent absolutely. Then system (1) is of class (S).

Proof. In system (1) make now the change of variables

$$x(t) = y(t) + Y_1(t)y(t) + Y_2(t)y(t).$$

where matrices $Y_1(t)$, $Y_2(t)$ are we shall choose later. We obtain

$$(I + Y_1(\sigma(t)) + Y_2(\sigma(t))y(t)^\Delta + (Y_1^\Delta(t))y(t) + (Y_2^\Delta(t))y(t) = (A(t) + A(t)Y_1(t) + A(t)Y_2(t))y(t) \quad (9)$$

Let

$$Y_1^\Delta(t) = A(t), \quad Y_2^\Delta(t) = A(t)Y_1(t).$$

Then

$$Y_1(t) = - \int_t^\infty A(s)\Delta s, \quad Y_2(t) = \int_t^\infty A(s) \left(\int_{\tau=s}^\infty A(\tau)\Delta\tau \right) \Delta s \quad (10)$$

From the formulas (10) follows that $Y_1(\sigma(t))$, $Y_2(\sigma(t))$ are converges to 0 as $t \rightarrow \infty$. Therefore matrices $I + Y_1(\sigma(t))$, $I + Y_2(\sigma(t))$ have a bounded converse for sufficiently large t . Hence system (9) can be written in following form

$$y^\Delta(t) = (I + Y_1(\sigma(t)) + Y_2(\sigma(t)))^{-1} A(t) Y_2(t) y(t). \quad (11)$$

From conditions theorem 3 follows that right part of system (6) satisfy conditions of theorem 1.

Generally, if integrals

$$\begin{aligned} & \int_a^\infty A(t)\Delta t, \quad \int_a^\infty A(t) \left(\int_{s=t}^\infty A(s)\Delta s \right) \Delta t, \dots, \\ & \int_a^\infty A(t_1) \left(\int_{t_2=t_1}^\infty A(t_2)\Delta t_2 \cdots \int_{t_k=t_{k-1}}^\infty A(t_k)\Delta t_k \right) \Delta t_1 \end{aligned}$$

are convergent and integral

$$\int_a^\infty A(t_1) \int_{t_2=t_1}^\infty A(t_2)\Delta t_2 \cdots \int_{t_{k+1}=t_k}^\infty A(t_{k+1})\Delta t_{k+1}) \Delta t_1$$

is convergent absolutely, then system (1) is of class (S).

Consider now the nonlinear system of difference equations

$$x^\Delta(t) = f(t, x(t)), \quad t \in T, \quad x \in R^N. \quad (12)$$

Theorem 4. Let $f(t, x)$ be defined for $t \in T$ $|x| < \delta$ ($\leq \infty$) and satisfy inequality

$$|f(t, x)| \leq K(t)|x|,$$

where

$$\int_a^\infty K(t)\Delta t < \infty.$$

If $|x_0|$ is sufficiently small, let us say

$$|x_0|e_K(t, t_0) < \delta, \quad (13)$$

Then for a solution $x(t)$ of (12) satisfying $x(a) = x_0$ exists

$$x_\infty = \lim_{t \rightarrow \infty} x(t)$$

and $x_\infty \neq 0$ unless $x(t) \equiv 0$.

Proof. The solutions of system (12) are represented in the form

$$x(t) = x(a) + \int_a^t f(s, x(s))\Delta s. \quad (14)$$

If $x(a) = x_0$ satisfies (14), then from (13) and conditions of theorem 4 follows that

$$|x(t)| \leq [|x(a)| + \int_a^t K(s)|x(s)|\Delta s].$$

The Gronwall's inequality gives

$$|x(t)| \leq |x(a)e_K(t, t_0)|.$$

Consequently all solutions of system (11) is bounded. Let $t_2 > t_1$. Let's evaluate norm $|x(t_2) - x(t_1)|$:

$$|x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} K(s)|x(s)|\Delta s. \quad (15)$$

From inequality (4) follows that $|x(t_2) - x(t_1)|$ can be made as much as small if to take t_1 big enough, Therefore $|x(t)|$ converges to the finite limits as $t \rightarrow \infty$. From Gronwall's inequality follows

$$|x(t)| \geq |x(s)|(e_K(t, s))^{-1}, \quad t, s \in T. \quad (16)$$

The inequality (16) implies that $x_\infty \neq 0$ unless $x(n) \equiv 0$. This proves theorem 4.

Let $T = \mathcal{R}$. Then the theorem 2 together with the remark 1 return to the results Wintner [1].

Let $T = \mathcal{Z}$. We consider a discrete adiabatic oscillator

$$x(n+2) - (2 \cos \alpha)x(n+1) + (1 + g(n))x(n) = 0, \quad (17)$$

where $0 < \alpha < \pi$, $g(n) \rightarrow 0$ as $n \rightarrow \infty$. For $g(n) \equiv 0$ the equation (16) has the form

$$x(n) = C_1 \cos n\alpha + C_2 \sin n\alpha.$$

We convert (16) into a system of equations by introducing new variables $C_1(n)$, $C_2(n)$

$$x(n) = C_1(n) \cos n\alpha + C_2(n) \sin n\alpha,$$

$$x(n+1) = C_1(n) \cos(n+1)\alpha + C_2(n) \sin(n+1)\alpha.$$

We obtain the system

$$\Delta u(n) = B(n)g(n)u(n),$$

where $\Delta u_1(n) = C_1(n+1) - C_1(n)$, $\Delta u_2(n) = C_2(n+1) - C_2(n)$. The matrix $B(n)$ has the form

$$B(n) = \frac{1}{\sin \alpha} (A_0 + A_1(n)),$$

where

$$A_0 = \begin{pmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{pmatrix}, \quad A_1(n) = \begin{pmatrix} \sin(2n+1)\alpha & -\cos(2n+1)\alpha \\ -\cos(2n+1)\alpha & -\sin(2n+1)\alpha \end{pmatrix}.$$

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